MTH 265. LN2C Week 2 Friday Lecture Notes.
Comparison Test and Limit Comparison Tests.

Proposition 1. Comparison Test (CT). Let $\sum a_n$ and $\sum b_n$ be series with positive terms.

① If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is divergent.

② If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is convergent.

Proof Sketch. We can get this result by comparing the sequence of partial sums S_n.

Let (S_n) and (P_n) be the sequence of partial sums of $\sum a_n$ and $\sum b_n$ respectively.

Assume that a_n and b_n are positive for all n. Then, (S_n) and (P_n) are both increasing sequences.

① If $\sum b_n$ is divergent, $\lim_{n\to\infty} P_n = \infty$.

Assuming $a_n \geq b_n$, then $\lim_{n\to\infty} S_n = \infty$ since $S_n \geq P_n$ for all n.

② Assume that $\sum b_n$ is convergent and let $B = \sum b_n = \lim_{n\to\infty} P_n$.

There's a theorem called the Monotone Convergence Theorem (MCT) that says if a sequence is increasing and bounded above, then the sequence converges.

Since (P_n) is increasing, $P_n \leq B$ for all n. Assuming $a_n \leq b_n$, $S_n \leq P_n \leq B$ for all n.

Therefore, (S_n) is bounded. From earlier, S_n is increasing. By MCT, $\lim_{n\to\infty} S_n$ exists.

$\therefore \sum a_n$ is convergent.

Note: We typically use p-series or geometric series for Comparison Tests.

Example 1.1. Determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$.

For $n \geq 1$: $2n^2+4n+3 \geq 2n^2 > 0$. Then, $0 < \dfrac{1}{2n^2+4n+3} \leq \dfrac{1}{2n^2}$ and $\dfrac{5}{2n^2+4n+3} \leq \left(\dfrac{5}{2}\right)\dfrac{1}{n^2}$ (*)

We know that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges since it's a p-series with $p=2>1$. Then, $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5}{2}\right)\frac{1}{n^2}$ converges.

By the Comparison Test with (*), $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges.

Example 1.2. Determine the convergence of $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-2}$.

Since $n \geq 3$: $n-2>0$. Then, $n > n-2 > 0$ and $0 < \frac{1}{n} < \frac{1}{n-2}$;

Since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$ is divergent, $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n-2}$ is divergent by the Comparison Test.

Example 1.3. Determine the convergence of $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$;

For $n \geq 3$: $n > e \approx 2.718$; Since $\ln(x)$ is increasing, $\ln(n) > \ln(e) = 1$;

Since n is positive, $\frac{\ln(n)}{n} > \frac{1}{n} > 0$; We know that $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges. By CT, $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$ diverges.

Therefore, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ diverges.

Non-Example 1.4. Determine the conv. of $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 20n}$;

We want to compare this to $\sum_{n=1}^{\infty} \frac{1}{3n^2}$ but for $n \geq 1$: $3n^2 > 3n^2 - 20n$;

For $n \geq 7$: both $3n^2$ and $3n^2 - 20n$ are positive. So, $\frac{1}{3n^2} < \frac{1}{3n^2 - 20n}$;

We can't use the Comparison Test here.

Proposition 2. The Limit Comparison Test (LCT). Let $\sum a_n$ and $\sum b_n$ be series with positive terms.

If $\lim_{n \to \infty} \frac{a_n}{b_n}$ is positive and finite, then $\sum a_n$ and $\sum b_n$ both converge or they both diverge.

Proof Sketch. Since a_n and b_n are positive for all n, $\frac{a_n}{b_n}$ is defined and positive for all n.

Assume $\lim_{n \to \infty} \frac{a_n}{b_n}$ exists and is positive. Let $c = \lim_{n \to \infty} \frac{a_n}{b_n}$.

Since $c > 0$, there exists $m, M \in \mathbb{R}$ positive such that $m < c < M$. e.g. $m = \frac{1}{2}c$ and $M = 2c$;

By definition of limit, for some tail sequence $\left(\frac{a_n}{b_n}\right)_{n=N}^{\infty}$ of $\left(\frac{a_n}{b_n}\right)$: $m < \frac{a_n}{b_n} < M$ for all $n \geq N$.

Then, $mb_n < a_n < Mb_n$. Consider the series $\sum mb_n$ and $\sum Mb_n$.

① If $\sum b_n$ is divergent, $\sum mb_n$ is divergent.
With $mb_n < a_n$ for all $n \geq N$, $\sum a_n$ diverges by the Comparison Test.

② If $\sum b_n$ is convergent, $\sum Mb_n$ is convergent.
With $a_n < Mb_n$ for all $n \geq N$, $\sum a_n$ converges by the Comparison Test.

Example 2.1. Determine the convergence of $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 20n}$;

Let $a_n = (3n^2 - 20n)^{-1}$ and let $b_n = n^{-2}$. We know that $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent as a p-series with $p > 1$.

Then, $\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \lim_{n \to \infty} \left(\frac{(3n^2 - 20n)^{-1}}{n^{-2}}\right) = \lim_{n \to \infty} \left(\frac{n^2}{3n^2 - 20n}\right) = \frac{1}{3}$;

Since $0 < \frac{1}{3} < \infty$, $\sum_{n=1}^{\infty} \frac{1}{3n^2 - 20n}$ converges by the LCT.

Example 2.2. Determine the convergence of $\sum_{n=0}^{\infty} \frac{1}{2^n-1}$;

Let $a_n = (2^n-1)^{-1}$ and $b_n = (\frac{1}{2})^n = (2^n)^{-1}$.
Observe that $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} (\frac{1}{2})^n$ converges as a geometric series with $|r| < 1$.
Then, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{(2^n-1)^{-1}}{(2^n)^{-1}}\right) = \lim_{n\to\infty}\left(\frac{2^n}{2^n+1}\right) = \lim_{n\to\infty}(1^n) = 1$.
Since $0 < 1 < \infty$, $\sum_{n=0}^{\infty} \frac{1}{2^n-1}$ converges by the LCT.

Non-example 2.3. Determine the convergence of $\sum_{n=5}^{\infty} \frac{1}{n^4+2n-10}$;

The following tests are invalid:

① Let $b_n = \frac{1}{-n^4}$; Invalid since b_n is not positive for all n.

② Let $b_n = \frac{1}{n^2}$; Invalid since $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{n^2}{n^4+2n-10}\right) = 0$ ← This has to be positive.

③ Let $b_n = \frac{1}{n^5}$; Invalid since $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{n^5}{n^4+2n-10}\right) = \infty$ ← The limit does not exist.

Non-example 2.4. The Comparison Test and the Limit Comparison Test cannot be applied to the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n!}$ since a_{2n+1} is negative for all n.

Example 2.5. Determine the convergence of $\sum_{n=3}^{\infty} \frac{\sqrt{n}+1}{n^2-5n+1}$.

Let $a_n = \frac{\sqrt{n}+1}{n^2-5n+1}$; Observe that for $n \geq 3$, a_n is positive.
Also, as $n \to \infty$, $\frac{\sqrt{n}+1}{n^2-5n+1} \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$, i.e. the term x^p with $p \geq 1$ and p maximal dominates as $n \to \infty$.
Let $b_n = \frac{\sqrt{n}}{n^2} = n^{-\frac{3}{2}}$; Then, $\sum_{n=3}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ converges as a p-series with $p > 1$.
Then, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{n}+1}{n^2-5n+1}\cdot\frac{n^2}{\sqrt{n}}\right) = \lim_{n\to\infty}\left(\frac{\sqrt{n}}{n^2}\cdot\frac{n^2}{\sqrt{n}}\right) = 1 > 0$.
By the Limit Comparison Test, $\sum_{n=3}^{\infty} \frac{\sqrt{n}+1}{n^2-5n+1}$ converges.

Proposition 3. Let $p(x)$ be any polynomial in x. Let $a > 0$.
Then, there exists $N \in \mathbb{Z}$ such that for all $n \geq N$: $n! > p(n)$. Similarly for $n! > a^{p(n)}$.
Furthermore, $\lim_{n\to\infty}\left(\frac{p(n)}{n!}\right) = 0$ and $\lim\left(\frac{a^{p(n)}}{n!}\right) = 0$.

This tells us that the Comparison Test and the Limit Comparison Test can't be used for $\sum_{n=1}^{\infty} \frac{1}{n!}$ against p-series or geometric series.